

# THE CONVEX HULL OF A VARIETY

KRISTIAN RANESTAD AND BERND STURMFELS

**ABSTRACT.** We present a characterization, in terms of projective biduality, for the hyper-surfaces appearing in the boundary of the convex hull of a compact real algebraic variety.

## 1. FORMULA FOR THE ALGEBRAIC BOUNDARY

Convex algebraic geometry is concerned with the algebraic study of convex sets that arise in polynomial optimization. One topic of recent interest is the convex hull  $\text{conv}(C)$  of a compact algebraic curve  $C$  in  $\mathbb{R}^n$ . Various authors have studied semidefinite representations [11, 18], facial structure [17, 23], and volume estimates [3, 19] for such convex bodies. In [14] we characterized the boundary geometry of  $\text{conv}(C)$  when  $n = 3$ . The boundary is formed by the edge surface and the tritangent planes, the degrees of which we computed in [14, Theorem 2.1]. Here, we extend our approach to varieties of any dimension in any  $\mathbb{R}^n$ .

Throughout this paper, we let  $X$  denote a compact algebraic variety in  $\mathbb{R}^n$  which affinely spans  $\mathbb{R}^n$ . We write  $\bar{X}$  for the Zariski closure of  $X$  in complex projective space  $\mathbb{CP}^n$ . Later we may add further hypotheses on  $X$ , e.g., that the complex variety  $\bar{X}$  be smooth or irreducible.

The convex hull  $P = \text{conv}(X)$  of  $X$  is an  $n$ -dimensional compact convex semialgebraic subset of  $\mathbb{R}^n$ . We are interested in the boundary  $\partial P$  of  $P$ . Basic results in convexity [10, Chapter 5] and real algebraic geometry [4, Section 2.8] ensure that  $\partial P$  is a semialgebraic set of pure dimension  $n - 1$ . The singularity structure of this boundary has been studied by S.D. Sedykh [20, 21]. Our object of interest is the *algebraic boundary*  $\partial_a P$ , by which we mean the Zariski closure of  $\partial P$  in  $\mathbb{CP}^n$ . Thus  $\partial_a P$  is a closed subvariety in  $\mathbb{CP}^n$  of pure dimension  $n - 1$ . We represent  $\partial_a P$  by the polynomial in  $\mathbb{R}[x_1, \dots, x_n]$  that vanishes on  $\partial P$ . This polynomial is unique up to a multiplicative constant as we require it to be squarefree. Our ultimate goal is to compute the polynomial representing the algebraic boundary  $\partial_a P$ .

We write  $X^*$  for the projectively dual variety to  $\bar{X}$ . The dual variety  $X^*$  lives in the dual projective space  $(\mathbb{CP}^n)^\vee$ . It is the Zariski closure of the set of all hyperplanes that are tangent to  $\bar{X}$  at a regular point. According to the *Biduality Theorem* of projective geometry, we have  $(X^*)^* = \bar{X}$ . We refer to [7, §I.1.3] for a proof of this important result.

For any positive integer  $k$  we let  $X^{[k]}$  denote the Zariski closure in  $(\mathbb{CP}^n)^\vee$  of the set of all hyperplanes that are tangent to  $\bar{X}$  at  $k$  regular points that span a  $(k-1)$ -plane. Thus  $X^{[1]} = X^*$  is the dual variety. We consider the following nested chain of algebraic varieties:

$$X^{[n]} \subseteq \dots \subseteq X^{[2]} \subseteq X^{[1]} \subseteq (\mathbb{CP}^n)^\vee.$$

Our objects of interest is the dual variety, back in  $\mathbb{CP}^n$ , to any  $X^{[k]}$  appearing in this chain.

To avoid anomalies, we make the assumption that only finitely many hyperplanes are tangent to  $\bar{X}$  at infinitely many points. Without this assumption, the relevant projective geometry is much more subtle, as seen in the recent work of Abuaf [1]. With this assumption, for small values of  $k$ , the dual variety  $(X^{[k]})^*$  equals the  $k$ -th secant variety of  $X$ , the closure of the union of all  $(k-1)$ -dimensional linear spaces that intersect  $X$  in at least  $k$  points.

The codimension of this secant variety is at least 2 if  $k \leq \lfloor \frac{n}{\dim(X)+1} \rfloor$ . Let  $r(X)$  be the minimal integer  $k$  such that the  $k$ -th secant variety of  $X$  has dimension at least  $n-1$ . Thus we have  $r(X) \geq \lceil \frac{n}{\dim(X)+1} \rceil$ . The inequality  $k \geq r(X)$  is necessary for  $(X^{[k]})^*$  to be a hypersurface. The main result in this article is the following formula for the convex hull.

**Theorem 1.1.** *Let  $X$  be a smooth and compact real algebraic variety that affinely spans  $\mathbb{R}^n$ , and assume that only finitely many hyperplanes in  $\mathbb{CP}^n$  are tangent to the corresponding projective variety  $\bar{X}$  at infinitely many points. The algebraic boundary of its convex hull,  $P = \text{conv}(X)$ , is computed by biduality as follows:*

$$(1.1) \quad \partial_a P \subseteq \bigcup_{k=r(X)}^n (X^{[k]})^*.$$

*In particular, every irreducible component of  $\partial_a P$  is a component of  $(X^{[k]})^*$  for some  $k$ .*

Since  $\partial_a P$  is a hypersurface, at least one of the  $(X^{[k]})^*$  must be a hypersurface. However, others may have higher codimension and these can be removed from the union. The reason for possibly not having equality in (1.1) is that some hypersurface component in  $(X^{[k]})^*$  may have no real points, or its real points may be disjoint from the boundary of  $P = \text{conv}(X)$ . Such components must also be removed when we compute the algebraic boundary  $\partial_a P$ .

When the inclusion  $X^{[k]} \subseteq X^{[k-1]}$  is proper, the former is part of the singular locus of the latter. In particular  $X^{[k]}$  is in general part of the  $k$ -tuple locus of the dual variety  $X^{[1]} = X^*$ . However, the singular locus of  $X^*$  will have further components. For example, the dual variety of a curve or surface in  $\mathbb{CP}^3$  has a cuspidal edge defined, respectively, by the osculating planes to the curve, and by planes that intersect the surface in a cuspidal curve.

Our presentation is organized as follows. In Section 2 we discuss a range of examples which illustrate the formula (1.1). The proof of Theorem 1.1 is given in Section 3. We also examine the case when  $X$  is not smooth, and we extend Theorem 1.1 to varieties whose real singularities are isolated. Section 4 features additional examples. These highlight the need to develop better symbolic and numerical tools for evaluating the right hand side of (1.1).

## 2. FIRST EXAMPLES

**2.1. Polytopes.** Our first example is the case of finite varieties, when  $\dim(X) = 0$ . Here  $P = \text{conv}(X)$  is a full-dimensional convex polytope in  $\mathbb{R}^n$ , and its algebraic boundary  $\partial_a P$  is the Zariski closure of the union of all facets of  $P$ . The formula (1.1) specializes to

$$\partial_a P \subseteq (X^{[n]})^*.$$

Indeed,  $X^{[n]} \subset (\mathbb{CP}^n)^\vee$  is the finite set of hyperplanes that are spanned by  $n$  affinely independent points in  $X$ . Typically, this includes hyperplanes that do not support  $\partial P$ , and these should be removed when passing from  $(X^{[n]})^*$  to  $\partial_a P$ . It is important to note that the Zariski closure, used in our definition of the algebraic boundary  $\partial_a P$ , depends on the field  $K \subseteq \mathbb{R}$  we are working over. If we take  $K = \mathbb{R}$  then  $\partial_a P$  is precisely the union of the facet hyperplanes of  $P$ . However, if  $K$  is the field of definition of  $X$ , say  $K = \mathbb{Q}$ , then  $\partial_a P$  usually contains additional hyperplanes that are Galois conjugate to the facet hyperplanes.

Here is a tiny example that illustrates this arithmetic subtlety. Let  $n = 1$  and take  $X$  to be the variety of the univariate polynomial  $x^5 - 3x + 1$ . This polynomial is irreducible over  $\mathbb{Q}$  and has three real roots. The smallest root is  $\alpha = -1.3888\dots$  and the largest root is  $\beta = 1.2146\dots$ . Clearly,  $P = \text{conv}(X)$  is the line segment  $[\alpha, \beta]$  in  $\mathbb{R}^1$ . If we take  $K = \mathbb{R}$  then  $\partial_a P = \{\alpha, \beta\}$ , but if we take  $K = \mathbb{Q}$  then  $\partial_a P$  consists of all five complex roots of  $f(x)$ .

**2.2. Irreducible Curves.** Let  $n = 2$  and  $X$  an irreducible compact curve in  $\mathbb{R}^2$  of degree  $d \geq 2$ . Since  $X$  is a hypersurface, we have  $r(X) = 1$ . Suppose that the curve  $X$  does not bound a convex region in  $\mathbb{R}^2$ . The algebraic boundary of the convex set  $P = \text{conv}(X)$  consists of  $X$  and the union of all bitangent lines of  $X$ . In symbols,

$$\partial_a P \subseteq (X^{[1]})^* \cup (X^{[2]})^* = X \cup (X^{[2]})^*.$$

For a smooth curve  $X$  of degree  $d$ , the classical Plücker formulas imply that the number of (complex) bitangent lines equals  $(d-3)(d-2)d(d+3)/2$ . Hence,  $\partial_a P$  is a curve of degree

$$\deg(\partial_a P) \leq d + \frac{(d-3)(d-2)d(d+3)}{2}.$$

Next consider the case where  $n = 3$ ,  $\dim(X) = 1$ , and  $r(X) = 2$ . If  $X$  is irreducible then

$$\partial_a P \subseteq (X^{[2]})^* \cup (X^{[3]})^*.$$

The first piece  $(X^{[2]})^*$  is the *edge surface* of  $X$ , and the second piece  $(X^{[3]})^*$  is the union of all *tritangent planes*. For a detailed study of this situation, including pretty pictures of  $P$ , and a derivation of degree formulas for  $(X^{[2]})^*$  and  $(X^{[3]})^*$ , we refer to our earlier paper [14]. Further examples of space curves are found in Subsection 4.1 below and in [14, Section 4].

Sedykh and Shapiro [19] studied *convex curves*  $X \subset \mathbb{R}^n$  where  $n = 2r$  is even. Such a curve has the property that  $|X \cap H| \leq n$  for every real hyperplane  $H$ . The algebraic boundary of a convex curve is the hypersurface of all secant  $(r-1)$ -planes. In symbols,  $\partial_a P = (X^{[r]})^*$ .

**2.3. Surfaces in 3-Space.** Let  $X$  be a general smooth compact surface of degree  $d$  in  $\mathbb{R}^3$ . Confirming classical derivations by Cayley, Salmon and Zeuthen [16, p.313-320], work on enumerative geometry in the 1970s by Piene [13, p.231] and Vainsencher [22, p.414] establishes the following formulas for the degree of the curve  $X^{[2]}$ , its dual surface  $(X^{[2]})^*$ ,

and the finite set  $X^{[3]}$  in  $(\mathbb{CP}^3)^\vee$ :

$$\begin{aligned} \deg(X^{[2]}) &= \frac{d(d-1)(d-2)(d^3 - d^2 + d - 12)}{2}, \\ \deg((X^{[2]})^*) &= d(d-2)(d-3)(d^2 + 2d - 4), \\ \deg(X^{[3]}) &= \deg((X^{[3]})^*) \\ &= \frac{d^9 - 6d^8 + 15d^7 - 59d^6 + 204d^5 - 339d^4 + 770d^3 - 2056d^2 + 1920d}{6}. \end{aligned}$$

We can expect the degree of  $\partial_a P$  to be bounded above by  $d$  plus the sum of the last two expressions, since

$$\partial_a P \subseteq (X^{[1]})^* \cup (X^{[2]})^* \cup (X^{[3]})^* = X \cup (X^{[2]})^* \cup (X^{[3]})^*,$$

unless  $X$  is convex or otherwise special. For a numerical example consider the case  $d = 4$ , where we take  $X$  to be a compact but non-convex smooth quartic surface in  $\mathbb{R}^3$ . The above formulas reveal that the degree of the algebraic boundary  $\partial_a P$  can be as large as

$$\deg(X) + \deg((X^{[2]})^*) + \deg((X^{[3]})^*) = 4 + 160 + 3200 = 3364.$$

**2.4. Barvinok-Novik curve.** We examine the first non-trivial instance of the family of *Barvinok-Novik curves* studied in [6, 23]. This is the curve  $X \subset \mathbb{R}^4$  parametrically given by

$$(c_1, c_3, s_1, s_3) = (\cos(\theta), \cos(3\theta), \sin(\theta), \sin(3\theta)).$$

We change to complex coordinates by setting  $x_j = c_j + \sqrt{-1} \cdot s_j$  and  $\bar{x}_j = c_j - \sqrt{-1} \cdot s_j$ . The convex body  $P = \text{conv}(X)$  is the projection of the 6-dimensional *Hermitian spectrahedron*

$$\left\{ (c_1, c_2, c_3, s_1, s_2, s_3) \in \mathbb{R}^6 : \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ \bar{x}_1 & 1 & x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 & 1 & x_1 \\ \bar{x}_3 & \bar{x}_2 & \bar{x}_1 & 1 \end{pmatrix} \text{ is positive semidefinite} \right\}$$

under the linear map  $\mathbb{R}^6 \rightarrow \mathbb{R}^4$ ,  $(c_1, c_2, c_3, s_1, s_2, s_3) \mapsto (c_1, c_3, s_1, s_3)$ . The curve  $X$  is the projection of the curve in  $\mathbb{R}^6$  that consists of the above Toeplitz matrices that have rank 1.

The convex body  $P = \text{conv}(X)$  was studied in [17, Example 5.5]. It is the 4-dimensional representative of the *Barvinok-Novik orbitopes* (cf. [6, 23]). Its algebraic boundary equals

$$\partial_a P = (X^{[2]})^* \cup (X^{[3]})^*.$$

The threefold  $(X^{[2]})^*$  represents the 2-dimensional family of edges of  $P$ , while the threefold  $(X^{[3]})^*$  represents the 1-dimensional family of triangles in  $\partial P$ , both of which are described in [6, Thm. 4.1]; see also [23]. The defining polynomials of these two hypersurfaces in  $\mathbb{R}^4$  are

$$\begin{aligned} &\langle x_3^2 \bar{x}_1^6 - 2x_1^3 x_3 \bar{x}_1^3 \bar{x}_3 + x_1^6 \bar{x}_3^2 + 4x_1^3 \bar{x}_1^3 - 6x_1 x_3 \bar{x}_1^4 - 6x_1^4 \bar{x}_1 \bar{x}_3 + 12x_1^2 x_3 \bar{x}_1^2 \bar{x}_3 \\ &\quad - 2x_3^2 \bar{x}_1^3 \bar{x}_3 - 2x_1^3 x_3 \bar{x}_3^2 - 3x_1^2 \bar{x}_1^2 + 4x_3 \bar{x}_1^3 + 4x_1^3 \bar{x}_3 - 6x_1 x_3 \bar{x}_1 \bar{x}_3 + x_3^2 \bar{x}_3^2 \rangle \end{aligned} \quad \text{from } (X^{[2]})^*,$$

$$\text{and} \quad \langle x_3 \bar{x}_3 - 1 \rangle = \langle c_3^2 + s_3^2 - 1 \rangle \quad \text{from } (X^{[3]})^*.$$

Both of these threefolds are irreducible components of the ramification locus that arises when we project the hypersurface of singular Toeplitz matrices from  $\mathbb{R}^6$  into  $\mathbb{R}^4$  as above.

### 3. PROOF OF THE FORMULA

We turn to the proof of our biduality formula for the algebraic boundary of  $P = \text{conv}(X)$ .

*Proof of Theorem 1.1.* We first prove that the supporting hyperplane of any exposed face  $F$  of  $P$  lies in  $X^{[k]}$  for some  $k$ . Suppose that  $\dim(F) = k - 1$  and let  $L_F$  be the projective span of  $F$ . By Carathéodory's Theorem, every point of  $F$  lies in the convex hull of  $k$  distinct points on  $X$ . In particular, the  $(k - 1)$ -plane  $L_F$  intersects  $X$  in at least  $k$  points that span a  $(k - 1)$ -simplex in  $F$ . If  $H$  is a supporting hyperplane for  $F$ , then  $H$  contains  $F$  and is the boundary of a halfspace that contains  $X$ . Since  $X$  is smooth, the tangent plane to  $X$  at each point  $q \in X \cap F \subseteq X \cap H$  must therefore be contained in  $H$ . We conclude that  $[H] \in X^{[k]}$ .

Now, consider any irreducible hypersurface  $Y \subset \mathbb{CP}^n$  whose real locus has full-dimensional intersection with the boundary  $\partial P \subset \mathbb{R}^n$ . We need to show that  $Y$  is a component of  $(X^{[k]})^*$  for some  $k$ . In the next paragraph we give an overview of the proof that follows thereafter.

First, we shall identify the relevant number  $k = k_Y + 1$ , by the property that  $Y$  has a linear space of dimension  $k_Y$  through every point. In fact, we shall prove that  $Y$  contains a unique  $k_Y$ -plane through a general point of  $Y$ . Thus, at a general point, the hypersurface  $Y$  is locally a fibration. In particular, the general point in  $\partial P \cap Y$  lies in a  $k_Y$ -plane that intersects  $P$  along a  $k_Y$ -dimensional face. Subsequently, we will show that the supporting hyperplanes of these faces are tangent to  $Y$  along these  $k_Y$ -planes, before we prove that  $Y^* \subseteq X^{[k]}$ . From this, we shall finally conclude that  $Y$  is a component of  $(X^{[k]})^*$ .

Let  $q$  be a general smooth point in the  $(n - 1)$ -dimensional semialgebraic set  $\partial P \cap Y$ . Since the union of the exposed faces of  $P$  is dense in  $\partial P$ , there exists an exposed face  $F_q$  that has  $q$  in its relative interior. The hypersurface  $Y$  contains the boundary of  $P$  locally at  $q$ , and hence it contains the face  $F_q$ . Since  $Y$  is a variety, it contains the projective span  $L_{F_q}$  of the face  $F_q$ . Let  $k_Y = \dim(L_{F_q})$ . Since  $q$  is a general smooth point in  $\partial P \cap Y$ , the hypersurface  $Y$  contains a  $k_Y$ -plane through every point of  $Y$ . In fact, since  $F_q$  is an exposed face, it is the unique face through  $q$ , so  $Y$  contains a unique  $k_Y$ -plane through every general point.

Next, let  $H$  be a hyperplane that exposes the  $k_Y$ -dimensional face  $F_q$  of  $P$ . We will show that  $H$  coincides with the tangent hyperplane  $H_q$  to  $Y$  at  $q$ . As  $q$  is a general interior point in  $F_q$ , we then conclude that  $H$  is tangent to  $Y$  along the entire  $k_Y$ -plane  $L_{F_q}$ . The key to our argument is that  $H$  is assumed to be tangent to  $X$  at the points  $X \cap F_q$  that span  $L_{F_q}$ .

If  $Y = L_{F_q}$  is itself a hyperplane, there is nothing to prove, except to note that  $k_Y = n - 1$ , that  $H = H_q$ , and that  $Y^*$  is an isolated point in  $X^{[n]}$ . Otherwise, we compare  $H$  and the tangent plane  $H_q$  via a local parameterization of  $Y$  at  $q$ . Let  $k = k_Y + 1$  and  $m = \dim(X)$ , let  $p_1, \dots, p_k$  be points in  $X \cap F_q$  that affinely span  $F_q$ , and let

$$\gamma_i : t_i = (t_{i,1}, \dots, t_{i,m}) \mapsto (\gamma_{i,1}(t_i), \dots, \gamma_{i,n}(t_i)) \quad (\text{for } i = 1, \dots, k)$$

be local parameterizations of  $\bar{X}$  at the points  $p_i$ . The point  $q$  lies in the affine-linear span of the points  $p_i$ , so  $q = \sum_i^k a_i p_i$  for some real coefficients  $a_i$  with  $\sum a_i = 1$ . There may be polynomial relations in the local parameters  $t_i$  defining  $k$ -tuples of points in  $X$  whose affine-linear span lies in  $Y$ . These relations define a subvariety  $Z$  in the Cartesian product  $\bar{X}^k$  that contains the  $k$ -tuple  $(p_1, \dots, p_k)$ . A local parametrization of  $L_{F_q}$  at  $q$  has the form

$$\alpha : u = (u_1, \dots, u_{k_Y}) \mapsto (\alpha_1(u), \dots, \alpha_n(u))$$

with affine-linear functions  $\alpha_i$  in the  $u_i$ . Since  $Y$  is locally a fibration, the algebraic functions  $\gamma_i$  and  $\alpha$  provide a local parameterization of the complex variety  $Y$  at the point  $q$ :

$$\begin{aligned} \Gamma : \quad \mathbb{C}^{k_Y} \times Z &\rightarrow \mathbb{C}^n \\ (u, t_1, \dots, t_k) &\mapsto \alpha(u) + \sum_{i=1}^k a_i(\gamma_i(t_i)) + \epsilon(u, t_1, \dots, t_k) \end{aligned}$$

Here, the function  $\epsilon$  only contains terms of order at least two in the parameters. The tangent space  $H_q$  at  $q$  is spanned by the linear terms in the above parameterization. But these linear terms lie in the span of  $(\alpha_1(u), \dots, \alpha_n(u))$  and the linear terms in  $(\gamma_1, \dots, \gamma_k)$ . The former span  $L_{F_q}$ , while the latter span the tangent spaces to  $\bar{X}$  at each of the points  $p_i$ . So, by assumption they all lie in the hyperplane  $H$  that supports  $\partial P$  at  $F_q$ . Therefore, the hyperplane  $H$  must coincide with the tangent plane  $H_q$  to  $Y$  at  $q$ . Since  $q$  is a general point not just in  $Y$  but also in  $L_{F_q}$ , we conclude that  $H$  is tangent to  $Y$  along the entire plane  $L_{F_q}$ .

We have shown that the tangent hyperplanes to  $Y$  are constant along the  $k_Y$ -planes contained in  $Y$ . This implies that the dimension of the dual variety  $Y^*$  is equal to  $n - k$  where  $k = k_Y + 1$ . Locally around the point  $q$ , these tangent hyperplanes support faces of dimension  $k_Y = k - 1$  the convex body  $P$ . This ensures that the inclusion  $Y^* \subseteq X^{[k]}$  holds.

We next claim that  $Y^*$  is in fact an irreducible component of the variety  $X^{[k]}$ . This will be a consequence of the following general fact which we record as a lemma.

**Lemma 3.1.** *Every irreducible component of  $X^{[k]}$  has dimension at most  $n - k$ .*

*Proof.* Let  $W \subseteq X^{[k]}$  be a component, and let  $k_W$  be the minimal  $l$  such that  $W$  is not contained in  $X^{[l+1]}$ . Then  $k_W \geq k$  and  $W$  is a component of  $X^{[k_W]}$ .

Let  $CX \subset \mathbb{CP}^n \times (\mathbb{CP}^n)^\vee$  be the conormal variety of  $\bar{X}$ , the closure of the set of pairs  $(p, [H]) \in \mathbb{CP}^n \times (\mathbb{CP}^n)^\vee$  such that the hyperplane  $H$  is tangent at the smooth point  $p \in X$ . It has dimension  $n - 1$ . By assumption, the projection  $\rho : CX \rightarrow (\mathbb{CP}^n)^\vee$  into the dual space has only finitely many infinite fibers. Therefore  $X^* = \rho(CX)$  is a hypersurface and  $W$  is part of its  $k_W$ -tuple locus. If  $[H]$  is a general point in  $W$ , then  $X^*$  has at least  $k_W$  branches at  $[H]$ . Let  $(p_1, [H]), \dots, (p_{k_W}, [H])$  be smooth points in  $CX$  in the fiber over  $[H]$ , such that  $p_1, \dots, p_{k_W}$  are linear independent points on  $X$ . Consider the tangent spaces  $T_1, \dots, T_{k_W}$  to  $CX$  at these points, and let  $U_i = \rho_T(T_i)$ ,  $i = 1, \dots, k_W$  be the corresponding linear spaces in the tangent cone to  $X^*$  at  $[H]$ , where  $\rho_T$  is the map induced by  $\rho$  on tangent spaces. Then the intersection  $U_1 \cap \dots \cap U_{k_W}$  contains the tangent space to  $W$  at  $[H]$ . But  $p_i \in U_i^\perp$ , so the orthogonal complement of the intersection satisfies

$$(U_1 \cap \dots \cap U_{k_W})^\perp = \text{span}(U_1^\perp \cup \dots \cup U_{k_W}^\perp) \supseteq \text{span}(p_1, \dots, p_{k_W}).$$

We conclude that the plane  $U_1 \cap \cdots \cap U_{k_W}$  has codimension at least  $k_W$  at  $[H]$ , and therefore the variety  $W$  has codimension at least  $k_W$  in  $(\mathbb{CP}^n)^\vee$ . Since  $k_W \geq k$  the lemma follows.  $\square$

At this point, we are pretty much done. To recap, recall that we have shown  $Y^* \subseteq X^{[k]}$ ,  $\dim(Y^*) = n - k$  and  $\dim(X^{[k]}) \leq n - k$ . If  $X^{[k]}$  is irreducible, then we have  $Y^* = X^{[k]}$  and  $Y = (X^{[k]})^*$  follows. Otherwise, if  $X^{[k]}$  has several components, then its dual  $(X^{[k]})^*$  is the union of the dual varieties of each component. One of these components is  $Y$ , and hence  $Y^*$  is a component of  $(X^{[k]})^*$ . Therefore, the formula (1.1) in Theorem 1.1 is indeed true.  $\square$

Theorem 1.1 extends in a straightforward manner to reduced and reducible compact real algebraic sets with isolated singularities. A colorful picture of a trigonometric space curve  $X$  with a singularity on the boundary of  $P = \text{conv}(X)$  is shown in [15, Figure 6]. Also, in Subsection 4.1 below we shall examine a reducible space curve with isolated singularities with the property that some (finitely many) hyperplanes that are tangent at infinitely many points.

Let  $X \subset \mathbb{R}^n$  be a finite union of compact varieties, and assume that  $X$  has only isolated singularities. As before, we write  $\bar{X}$  be its Zariski closure in  $\mathbb{CP}^n$ . For any positive integer  $k$  we now take  $X^{[k]}$  to be the Zariski closure in  $(\mathbb{CP}^n)^\vee$  of the set of all hyperplanes that are tangent to  $\bar{X}$  at  $k - s$  regular points and pass through  $s$  singularities on  $X$ , for some  $s$ , such that the  $(k - s) + s = k$  points span a  $(k - 1)$ -plane. Thus  $X^{[1]}$  contains the dual variety, but, in addition, it also contains a hyperplane for each isolated singularity of  $X$ . We consider, as above, the nested chain of projective varieties

$$X^{[n]} \subseteq \cdots \subseteq X^{[2]} \subseteq X^{[1]} \subseteq (\mathbb{CP}^n)^\vee.$$

The algebraic boundary of  $P = \text{conv}(X)$  is dual to the various  $X^{[k]}$  appearing in this chain:

**Theorem 3.2.** *Let  $X$  be a finite union of compact real algebraic varieties that affinely spans  $\mathbb{R}^n$ , and assume that  $X$  has only isolated singularities and that only finitely many hyperplanes in  $\mathbb{CP}^n$  are tangent to  $\bar{X}$  at infinitely many points. The algebraic boundary of its convex hull,  $P = \text{conv}(X)$ , is computed by biduality using the same formula (1.1) as in Theorem 1.1. In particular, every irreducible component of  $\partial_a P$  is a component of  $(X^{[k]})^*$  for some  $k$ .*

*Proof.* Following the argument of the proof of Theorem 1.1, we first note that a hyperplane  $H$  that supports a  $(k - 1)$ -dimensional face of  $P$  must intersect  $X$  in  $k$  points that span the face. Furthermore,  $H$  must be tangent to  $X$  at the smooth intersection points. Let  $Y$  be an irreducible component having full-dimensional intersection with the boundary  $\partial P$  of  $\text{conv}(X)$ , and let  $q$  be a general smooth point on  $\partial P \cap Y$ . In the notation of the above proof, a local parameterization of  $Y$  at  $q$  will involve singular points  $p_1, \dots, p_s$  and smooth points  $p_{s+1}, \dots, p_k$ . The  $k$ -tuples  $(p_1, \dots, p_k)$  of points whose linear span is contained in  $Y$  form a subvariety  $Z$  in the Cartesian product  $X^k$ . Since the singular points are isolated, we may assume that the restriction of  $Z$  to the first  $s$  factors is a point. The hypersurface  $Y$  is therefore a cone with vertex containing the  $s$  singular points. The tangent hyperplane to  $Y$  at  $q$  contains the vertex and the tangent spaces at the  $k - s$  smooth points, so it coincides

with the supporting hyperplane  $H$ . The latter part of the proof of Theorem 1.1 applies directly to arrive at the same conclusion.  $\square$

At present, we do not know how to extend our formula (1.1) for the algebraic boundary to the convex hull of a compact real variety  $X$  whose real singular locus has dimension  $\geq 1$ . Also, we do not yet know how to remove the hypothesis that only finitely many hyperplanes are tangent to  $\bar{X}$  at infinitely many points. This issue is related to the study of degeneracies in [1] and we hope that the techniques introduced in that paper will help for our problem.

#### 4. MORE EXAMPLES AND COMPUTATIONAL THOUGHTS

We further illustrate our formula for the algebraic boundary of the convex hull of a real variety with three concrete examples, starting with a curve that is reducible and singular.

**4.1. Circles and spheres in 3-Space.** Let  $n = 3$  and suppose that  $X = C_1 \cup C_2 \cup \dots \cup C_r$  is the reducible (and possibly singular) curve obtained by taking the union of a collection of  $r \geq 3$  sufficiently general circles  $C_i$  that lie in various planes in  $\mathbb{R}^3$ . We have

$$(4.1) \quad \partial_a P \subseteq (X^{[2]})^* \cup (X^{[3]})^*.$$

The surface  $(X^{[3]})^*$  is the union of planes that are tangent to three of the circles and planes spanned by the circles. The edge surface  $(X^{[2]})^*$  decomposes into quadratic surfaces, namely, its components are cylinders formed by stationary bisecant lines defined by pairs of circles.

For a concrete configuration, consider the convex hull of  $r = 4$  pairwise touching circles in 3-space. The surface  $(X^{[2]})^*$  is a union of six cylinders, each wrapped around two of the circles, while  $(X^{[3]})^*$  is the union of planes tangent to three of the circles (four of which contain the fourth circle). A picture of this 3-dimensional convex body  $P$  is shown in Figure 1. Its boundary consists of  $6 + (4 + 4) = 14$  distinct surface patches, corresponding to the pieces in (4.1), which holds with equality. There are six cylinders, four planes touching exactly three of the circles, and four planes containing one of the circles and touching the three others.

A nice symmetric representation of the curve  $X = C_1 \cup C_2 \cup C_3 \cup C_4$  is given by the ideal

$$\langle a c g t, a^2 + c^2 + g^2 + t^2 - 2ac - 2ag - 2at - 2cg - 2ct - 2gt \rangle,$$

where the variety of that ideal is to be taken inside the probability simplex

$$\Delta_3 = \{ (a, c, g, t) \in \mathbb{R}_{\geq 0}^4 : a + c + g + t = 1 \}.$$

The convex body  $P$  looks combinatorially like a 3-polytope with 18 vertices, 36 edges and 20 cells. Eight of the 20 cells are flat facets. First, there are the planes of the circles themselves. For instance, the facet in the plane  $t = 0$  is the disk  $\{a^2 + c^2 + g^2 \leq 2ac + 2ag + 2cg\}$  in the triangle  $\{a + c + g = 1\}$ . Second, there are four triangle facets, formed by the unique planes that are tangent to exactly three of the circles. The equations of these facet planes are

$$\begin{aligned} P_a &= -a + 2c + 2g + 2t, & P_c &= 2a - c + 2g + 2t, \\ P_g &= 2a + 2c - g + 2t, & P_t &= 2a + 2c + 2g - t. \end{aligned}$$



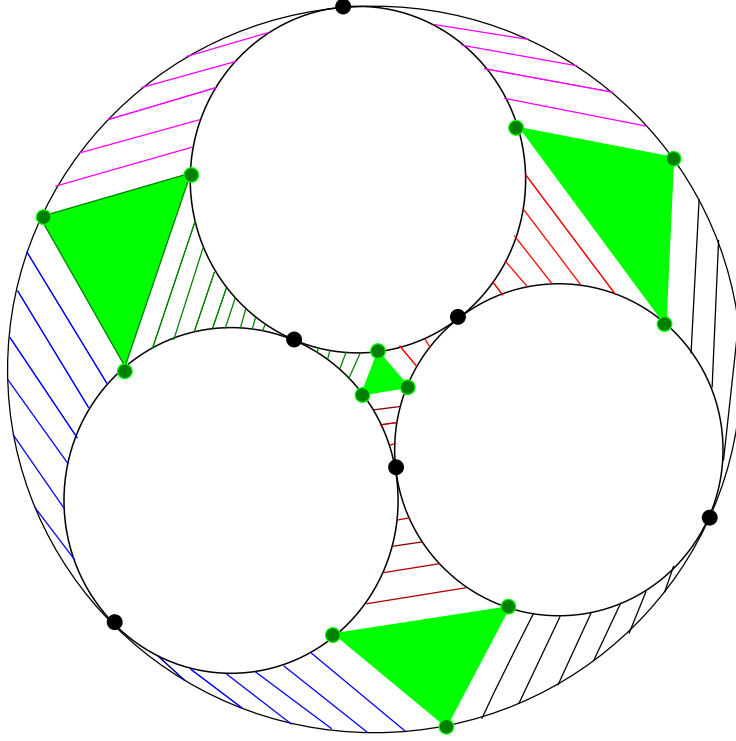
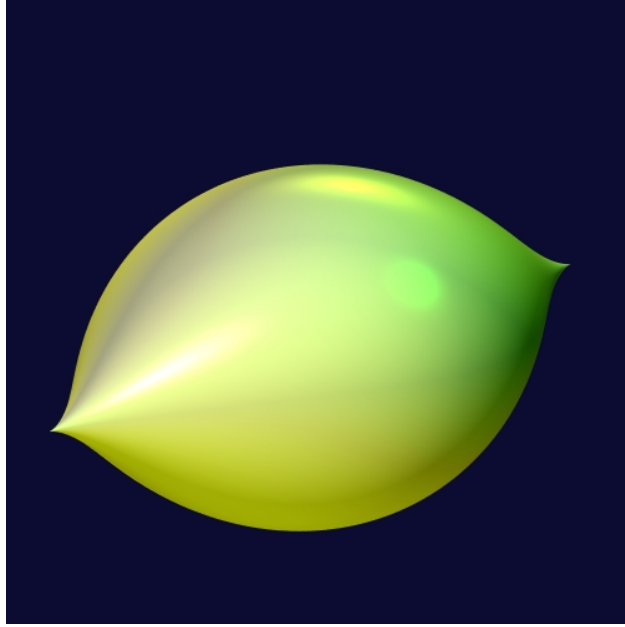


FIGURE 1. Schlegel diagram of the convex hull of four pairwise touching circles

The remaining 12 cells in  $\partial P$  are quadratic surface patches that arise from the pairwise convex hull of any two of the four circles. This results in 6 quadratic surfaces each of which contributes two triangular cells to the boundary. The equations of these six surfaces are

$$\begin{aligned} Q_{ac} &= a^2 + c^2 + g^2 + t^2 + 2(ac - ag - cg - at - ct - gt), \\ Q_{ag} &= a^2 + c^2 + g^2 + t^2 - 2(ac - ag + cg + at + ct + gt), \\ Q_{ag} &= a^2 + c^2 + g^2 + t^2 - 2(ac + ag - cg + at + ct + gt), \\ Q_{cg} &= a^2 + c^2 + g^2 + t^2 - 2(ac + ag + cg - at + ct + gt), \\ Q_{ct} &= a^2 + c^2 + g^2 + t^2 - 2(ac + ag + cg + at - ct + gt), \\ Q_{gt} &= a^2 + c^2 + g^2 + t^2 - 2(ac + ag + cg + at + ct - gt). \end{aligned}$$

Each circle is subdivided into six arcs of equal length. Three of the nodes arise from intersections with other circles, and the others are the intersections with the planes  $P_a, P_c, P_g, P_t$ . This accounts for all 18 vertices and 24 “edges” that are arcs. The other 12 edges of  $\partial P$  are true edges: they arise from the four triangles. These are shown in green in the *Schlegel diagram* of Figure 1. The 12 cells corresponding to the six quadratic surfaces are the 12 ruled cells in the diagram, and they come in pairs according to the six different colors. The six intersection points among the 4 circles are indicated by black dots, whereas the remaining twelve vertices correspond to the green dots which are vertices of our four green triangles.

FIGURE 2. The Zitrus surface  $x^2 + z^2 + (y^2 - 1)^3 = 0$ 

4.2. **Zitrus.** We have seen that the convex hull of algebraic surfaces in  $\mathbb{R}^3$  can have surfaces of very high degree in its boundary. For instance, if  $X$  is a general smooth surface of degree  $d = 6$  then the bitangent surface  $(X^{[2]})^*$  has degree 3168. On the other hand, that number can be expected to drop substantially for most singular surfaces. Let us consider the sextic

$$f(x, y, z) = x^2 + z^2 + (y^2 - 1)^3.$$

The surface  $X = V(f)$  in  $\mathbb{R}^3$  is taken from Herwig Hauser's beautiful *Gallery of Algebraic Surfaces*. The name given to that surface is *Zitrus*. It appears on page 42-43 of the catalog [9] of the exhibition *Imaginary*. For an electronic version see [www.freigeist.cc/gallery.html](http://www.freigeist.cc/gallery.html).

We choose affine coordinates  $(a, b, c)$  on the space of planes  $ax + by + cz + 1 = 0$  in  $\mathbb{R}^3$ . In these coordinates, the variety  $X^{[2]}$  is the union of two quadratic curves given by the ideal

$$\langle b + 1, 27a^2 + 27c^2 - 16 \rangle \cap \langle b - 1, 27a^2 + 27c^2 - 16 \rangle.$$

These curves parametrize the tangent planes that pass through one of the two singular points of the Zitrus. Each curve dualizes to a singular quadratic surface, and  $(X^{[2]})^*$  is given by

$$\langle 16x^2 - 27y^2 + 16z^2 + 54y - 27 \rangle \cup \langle 16x^2 - 27y^2 + 16z^2 - 54y - 27 \rangle.$$

The Zitrus  $X$  has no tritangent planes, so  $\partial_a P = X \cup (X^{[2]})^*$ , and we conclude that the algebraic boundary of the *convexified Zitrus*  $P = \text{conv}(X)$  has degree  $10 = 6 + 2 + 2$ .

We now perturb the polynomial  $f$  and consider the smooth surface  $\tilde{X} = V(\tilde{f})$  defined by

$$\tilde{f}(x, y, z) = x^2 + z^2 + (y^2 - 1)^3 - 1.$$

The curve of bitangent planes,  $\tilde{X}^{[2]}$ , has again two components. It is defined by the ideal

$$\langle b, a^2 + c^2 - 1 \rangle \cap \langle 90a^2b^2 - 96b^4 + 90b^2c^2 - 129a^2 + 128b^2 - 129c^2 + 48, \\ 135a^4 - 144b^4 + 270a^2c^2 + 135c^4 - 6a^2 + 272b^2 - 6c^2 - 48 \rangle.$$

The first curve dualizes to the cylinder  $\{x^2 + z^2 = 1\}$ . The other component of the boundary surface  $(\tilde{X}^{[2]})^*$  has degree 16. Its defining polynomial has 165 terms which start as follows:

$$16777216x^{16} - 169869312x^{14}y^2 + 1601372160x^{12}y^4 - 7081205760x^{10}y^6 + 26435102976x^8y^8 - \dots$$

**4.3. Grassmannian.** We consider the oriented Grassmannian  $X = \text{Gr}(2, 5)$  of oriented two-dimensional linear subspaces of  $\mathbb{R}^5$ . This is the 6-dimensional subvariety of  $\mathbb{R}^{10}$  defined by

$$\langle p_{12}^2 + p_{13}^2 + p_{14}^2 + p_{15}^2 + p_{23}^2 + p_{24}^2 + p_{25}^2 + p_{34}^2 + p_{35}^2 + p_{45}^2 - 1, p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}, \\ p_{12}p_{35} - p_{13}p_{25} + p_{15}p_{23}, p_{12}p_{45} - p_{14}p_{25} + p_{15}p_{24}, p_{13}p_{45} - p_{14}p_{35} + p_{15}p_{34}, p_{23}p_{45} - p_{24}p_{35} + p_{25}p_{34} \rangle.$$

Its convex hull  $P = \text{conv}(X)$  is a *Grassmann orbitope*, a class of convex bodies that are of interest to differential geometers. We refer to [12], [17, §7], and the references given therein. The determinant of the Hermitian matrix in the spectrahedral representation of  $P$  in [17, Theorem 7.3] has degree 8 and it factors into two quartic factors. Only one of these two factors is relevant for us, and we display it below. Namely, the algebraic boundary  $\partial_a P = (X^{[4]})^*$  is the irreducible hypersurface of degree 4 represented by the polynomial

$$\begin{aligned} & p_{12}^4 + p_{13}^4 + p_{14}^4 + p_{15}^4 + p_{23}^4 + p_{24}^4 + p_{25}^4 + p_{34}^4 + p_{35}^4 + p_{45}^4 \\ & + 2p_{12}^2p_{13}^2 + 2p_{12}^2p_{14}^2 + 2p_{13}^2p_{14}^2 + 2p_{12}^2p_{15}^2 + 2p_{13}^2p_{15}^2 + 2p_{14}^2p_{15}^2 + 2p_{12}^2p_{23}^2 + 2p_{13}^2p_{23}^2 - 2p_{14}^2p_{23}^2 \\ & - 2p_{15}^2p_{23}^2 + 2p_{12}^2p_{24}^2 - 2p_{13}^2p_{24}^2 + 2p_{14}^2p_{24}^2 - 2p_{15}^2p_{24}^2 + 2p_{23}^2p_{24}^2 + 2p_{12}^2p_{25}^2 - 2p_{13}^2p_{25}^2 - 2p_{14}^2p_{25}^2 \\ & + 2p_{15}^2p_{25}^2 + 2p_{23}^2p_{25}^2 + 2p_{24}^2p_{25}^2 - 2p_{12}^2p_{34}^2 + 2p_{13}^2p_{34}^2 + 2p_{14}^2p_{34}^2 - 2p_{15}^2p_{34}^2 + 2p_{23}^2p_{34}^2 + 2p_{24}^2p_{34}^2 \\ & - 2p_{25}^2p_{34}^2 - 2p_{12}^2p_{35}^2 + 2p_{13}^2p_{35}^2 - 2p_{14}^2p_{35}^2 + 2p_{15}^2p_{35}^2 + 2p_{23}^2p_{35}^2 - 2p_{24}^2p_{35}^2 + 2p_{25}^2p_{35}^2 + 2p_{34}^2p_{35}^2 \\ & - 2p_{12}^2p_{45}^2 - 2p_{13}^2p_{45}^2 + 2p_{14}^2p_{45}^2 + 2p_{15}^2p_{45}^2 - 2p_{23}^2p_{45}^2 + 2p_{24}^2p_{45}^2 + 2p_{25}^2p_{45}^2 + 2p_{34}^2p_{45}^2 + 2p_{35}^2p_{45}^2 \\ & + 8p_{13}p_{14}p_{23}p_{24} + 8p_{13}p_{15}p_{23}p_{25} + 8p_{14}p_{15}p_{24}p_{25} - 8p_{12}p_{14}p_{23}p_{34} + 8p_{12}p_{13}p_{24}p_{34} \\ & - 8p_{12}p_{15}p_{23}p_{35} + 8p_{12}p_{13}p_{25}p_{35} + 8p_{14}p_{15}p_{34}p_{35} + 8p_{24}p_{25}p_{34}p_{35} - 8p_{12}p_{15}p_{24}p_{45} \\ & + 8p_{12}p_{14}p_{25}p_{45} - 8p_{13}p_{15}p_{34}p_{45} - 8p_{23}p_{25}p_{34}p_{45} + 8p_{13}p_{14}p_{35}p_{45} + 8p_{23}p_{24}p_{35}p_{45} \\ & - 2p_{12}^2 - 2p_{13}^2 - 2p_{14}^2 - 2p_{15}^2 - 2p_{23}^2 - 2p_{24}^2 - 2p_{25}^2 - 2p_{34}^2 - 2p_{35}^2 - 2p_{45}^2 + 1. \end{aligned}$$

This hypersurface represents a 6-dimensional family of 3-dimensional facets of  $P$ . Each facet of  $P$  is a 3-dimensional ball. It meets the variety  $X$  in its boundary, which is a 2-sphere.

**4.4. Computing.** This paper raises the following algorithmic problem: given a projective variety  $X$ , either by its ideal or by a parametrization, how to compute the equations defining  $(X^{[k]})^*$  in practise? The passage from  $X$  to  $X^{[k]}$  can be phrased as an elimination problem in a fairly straightforward manner. In principle, we can use any Gröbner-based computer algebra system to perform that elimination task. However, in our experience, this approach only succeeds for tiny low-degree instances. Examples such as the Grassmannian in Subsection 4.3 appear to be out of reach for a general purpose implementations of our formula (1.1).

Even the first instance  $k = 1$ , which is the passage from a variety  $X$  to its dual variety  $X^*$ , poses a considerable challenge for current computational algebraic geometry software.

The case of plane curve is still relatively easy, and it has been addressed in the literature [5]. However, what we need here is the case when  $X$  is not a hypersurface but  $X^*$  is. The first interesting situation is that of a space curve  $X \subset \mathbb{CP}^3$ . Our computations for space curves, both here and in [14], were performed in `Macaulay2` [8], but, even with ad hoc tricks, they turned out to be more difficult than we had expected when we first embarked on our project.

Here is an illustration of the issue. Let  $X$  be the smooth sextic curve in  $\mathbb{CP}^3$  defined by

$$\langle x^2 + y^2 + z^2 + w^2, xyz - w^3 \rangle.$$

The following lines of `Macaulay2` code find the surface  $X^*$  in  $(\mathbb{CP}^3)^\vee$  that is dual to  $X$ :

```
S = QQ[x,y,z,w,X,Y,Z,W];
d = 4; pairing = first sum(d,i->(gens S)_i*(gens S)_{i+d});
makedual = I -> (e = codim I; J =
saturate(I + minors(e+1,submatrix(jacobian(I+ideal(pairing)),{0..d-1})),
minors(e,submatrix(jacobian(I),{0..d-1})));eliminate((gens S)_{0..d-1},J))
makedual ideal( x^2+y^2+z^2+w^2, x*y*z-w^3 );
```

This program runs for a few minutes and outputs a polynomial of degree 18 with 318 terms:

$$729x^{14}y^4 + 3861x^{12}y^6 + 7954x^{10}y^8 + 7954x^8y^{10} + 3861x^6y^{12} + 729x^4y^{14} + 1458x^{14}y^2z^2 + \dots$$

Projective duality tends to produce large equations, even on modestly sized input, and symbolic programs, like our little `Macaulay2` fragment above, will often fail to terminate.

One promising alternative line of attack is offered by numerical algebraic geometry [2]. Preliminary experiments by Jonathan Hauenstein demonstrate that the software `Bertini` can perform the transformations  $X \mapsto X^*$  and  $X \mapsto (X^{[k]})^*$  in a purely numerical manner.

Convex algebraic geometry requires the development of new specialized software tools, both symbolic and numeric, and integrated with optimization method. The advent of such new tools will make our formula (1.1) more practical for non-linear convex hull computations.

**Acknowledgments.** This project started at the Banff International Research Station (BIRS) during the workshop *Convex Algebraic Geometry* (February 14-18, 2010). We are grateful to BIRS. Angelica Cueto and Herwig Hauser kindly allowed us to use their respective Figures 1 and 2. We thank Roland Abuaf for his careful reading of the first version of this paper. Bernd Sturmfels was supported in part by NSF grant DMS-0757207.

## REFERENCES

- [1] R. Abuaf: Singularities of the projective dual variety, [arXiv:0901.1821](#).
- [2] D. Bates, J. Hauenstein, A. Sommese, and C. Wampler: Software for numerical algebraic geometry: a paradigm and progress towards its implementation, in *Software for Algebraic Geometry* (eds. M. Stillman, N. Takayama, J. Verschelde), IMA Volumes in Math.Appl., **148**, 1-14, 2008, Springer, New York.
- [3] A. Barvinok and G. Blekherman: Convex geometry of orbits. *Combinatorial and Computational Geometry*, 51–77, Math. Sci. Res. Inst. Publ., 52, Cambridge Univ. Press, Cambridge, 2005.
- [4] J. Bochnak, M. Coste, M.-F. Roy: *Géométrie Algébrique Réelle*, Ergebnisse der Mathematik und ihrer Grenzgebiete, bf 12, Springer, Berlin, 1987

- [5] D. Bouziane and M. El Kahoui: Computation of the dual of a plane projective curve, *J. Symbolic Comput.* **34** (2002) 105–117.
- [6] A. Barvinok and I. Novik: A centrally symmetric version of the cyclic polytope, *Discrete Comput. Geom.* **39** (2008) 76–99.
- [7] I.M. Gel’fand, M.M. Kapranov and A.V. Zelevinsky: *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, 1994.
- [8] D. Grayson and M. Stillman: Macaulay2, a software system for research in algebraic geometry, Available at <http://www.math.uiuc.edu/Macaulay2/>.
- [9] G.-M. Greuel and A.D. Matt (eds.): *Imaginary - Mit den Augen der Mathematik. Through the Eyes of Mathematics*, Mathematisches Forschungsinstitut Oberwolfach, 2008.
- [10] P. Gruber: *Convex and Discrete Geometry*, Grundlehren der Mathematischen Wissenschaften, **336**, Springer, Berlin, 2007
- [11] D. Henrion: Semidefinite representation of convex hulls of rational varieties, [arXiv:0901.1821](https://arxiv.org/abs/0901.1821).
- [12] F. Morgan: Area-minimizing surfaces, faces of Grassmannians, and calibrations, *Amer. Math. Monthly* **95** (1988) 813–822.
- [13] R. Piene: Some formulas for a surface in  $\mathbb{P}^3$ , in *Algebraic Geometry* (Proc. Sympos., Tromsø, 1977), pp. 196–235, Lecture Notes in Mathematics, **687**, Springer, Berlin, 1978.
- [14] K. Ranestad and B. Sturmfels: On the convex hull of a space curve, *Advances in Geometry*, to appear.
- [15] P. Rostalski and B. Sturmfels: Dualities in convex algebraic geometry, [arXiv:1006.4894](https://arxiv.org/abs/1006.4894).
- [16] G. Salmon: *Treatise on the Analytic Geometry of Three Dimensions*, A treatise on the analytic geometry of three dimensions. Revised by R. A. P. Rogers. 5th ed., Vol. 2, Hodges, Figgis and Co., Dublin, 1915, reprinted by Chelsea Publ. Co., New York, 1965.
- [17] R. Sanyal, F. Sottile and B. Sturmfels: Orbitopes, [arXiv:0911.5436](https://arxiv.org/abs/0911.5436).
- [18] C. Scheiderer: Convex hulls of curves of genus one, [arXiv:1003.4605](https://arxiv.org/abs/1003.4605).
- [19] V.D. Sedykh and B. Shapiro: On Young hulls of convex curves in  $\mathbb{R}^{2n}$ , *Journal of Geometry* **63** (1998), no. 1-2, 168–182
- [20] V.D. Sedykh, Singularities of convex hulls, *Sibirsk. Mat. Zh.* **24** (1983), no.3, 158–175; English transl. in *Siberian Math. J.* **24** (1983), no.3, 447–461
- [21] V.D. Sedykh, Stabilization of the singularities of convex hulls, *Mat. Sb.(N.S.)* **135** (1988), no.4, 514–519; English transl. in *Math. USSR-Sb.* **63** (1989), no.2, 499–505.
- [22] I. Vainsencher: Counting divisors with prescribed singularities, *Transactions AMS*, **267** (1981) 399–422.
- [23] C. Vinzant: Edges of the Barvinok-Novik orbitope, [arXiv:1003.4528](https://arxiv.org/abs/1003.4528).

KRISTIAN RANESTAD, MATEMATISK INSTITUTT, UNIVERSITETET I OSLO, PO BOX 1053, BLINDERN, NO-0316 OSLO, NORWAY

*E-mail address:* [ranestad@math.uio.no](mailto:ranestad@math.uio.no)

BERND STURMFELS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720, USA

*E-mail address:* [bernd@math.berkeley.edu](mailto:bernd@math.berkeley.edu)